ON INCLUSION RELATIONS FOR SPACES OF AUTOMORPHIC FORMS

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1. INTRODUCTION

Let Γ be a Fuchsian group, that is a discontinuous group of Moebius transformations of the unit disk D onto itself, and let F be a fundamental domain of Γ with area $\partial F = 0$. For $q = 1,2,\cdots$ and $1 \leqq p \leqq \infty$, let $A_q^p(\Gamma)$ denote the space of functions $g(z)$ analytic in D that satisfy

$$g(\phi(z))\phi'(z)^q = g(z) \qquad (\phi \in \Gamma) \tag{1.1}$$

and

$$\begin{aligned} &\iint_F (1-|z|^2)^{pq-2}|g(z)|^p \, dx\, dy < \infty && \text{if } 1 \leqq p < \infty, \\ &\sup_{z\in F} (1-|z|^2)^q |g(z)| < \infty && \text{if } p = \infty. \end{aligned} \tag{1.2}$$

The integral is independent of the choice of the fundamental domain F; the supremum is not changed if we replace F by D. The spaces $A_q^1(\Gamma)$ and $A_q^\infty(\Gamma)$ $(q = 2,3,\cdots)$ are of particular interest in the theory of Fuchsian and Kleinian groups [1],[2],[4].

Several authors (for instance [3],[8],[10],[6]) have considered the problem whether

$$A_q^p(\Gamma) \subset A_q^\infty(\Gamma) \qquad (1 \leqq p < \infty). \tag{1.3}$$

Rajeswara Rao [10] has shown that, for $q > 1$,

$$A_q^p(\Gamma) \subset A_q^\infty(\Gamma) \text{ for some } p < \infty \Rightarrow A_q^{p_1}(\Gamma) \subset A_q^{p_2}(\Gamma) \text{ for } 1 \leqq p_1 < p_2 \leqq \infty. \tag{1.4}$$

Hence it suffices to consider the case $p = 1$.

J. Lehner [5] has recently proved (1.3) for the case that there exists a constant $\gamma = \gamma(\Gamma) > 0$ such that

$$\inf_{z \in D} d(z,\phi(z)) \geqq \gamma \qquad \text{for all hyperbolic } \phi \in \Gamma,$$

where d denotes the non-euclidean distance. He uses A. Marden's results on universal properties of Fuchsian groups [7]. In particular, it follows that (1.3) holds if Γ is any subgroup of a finitely generated group.

We shall show that (1.3) is not true without some restriction on Γ.

THEOREM 1. *There exists a Fuchsian group* Γ *such that*

(1.5) $$A_1^2(\Gamma) \not\subset A_1^\infty(\Gamma)$$

and therefore that

(1.6) $$A_2^p(\Gamma) \not\subset A_2^\infty(\Gamma) \qquad (1 \leqq p < \infty).$$

To see that (1.5) implies (1.6) we choose a function $g \in A_1^2(\Gamma) \setminus A_1^\infty(\Gamma)$. Then $g^2 \in A_2^1(\Gamma) \setminus A_2^\infty(\Gamma)$, and (1.6) follows from (1.4). It is a pleasure to acknowledge conversations with A. Marden and L.Greenberg on this counterexample.

In the last section we establish a generally valid inclusion relation (with $q = 1$) in which $A_1^2(\Gamma)$ is replaced by a somewhat different space.

2. THE COUNTEREXAMPLE

The function $f(z)$ is called a Bloch function if it is analytic in D and satisfies

(2.1) $$\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$

We shall need the classical characterization in terms of schlicht disks on the Riemann image surface (see, e.g., [9]):

LEMMA. *An analytic function is not a Bloch function if and only if, for every* $\rho < +\infty$, *it maps some domain in* D *one-to-one onto a disk of radius* ρ.

It will be convenient to integrate condition (1.1) which, for $q = 1$, becomes $g(\phi(z))\phi'(z) = g(z)$. The function

$$f(z) = \int_0^z g(\zeta)\,d\zeta \qquad (z \in D) \tag{2.2}$$

is not, in general, automorphic but has periods $c(\phi)$ that satisfy

$$f(\phi(z)) = f(z) + c(\phi) \qquad (\phi \in \Gamma). \tag{2.3}$$

It follows from (1.2) and (2.1) that $A_1^\infty(\Gamma)$ consists of the derivatives of the Bloch functions for which (2.3) holds. Hence Theorem 1 is contained in the following theorem.

THEOREM 2. *There exists a Fuchsian group of the second kind and a non-Bloch function* $f(z)$ *satisfying (2.3) for which*

$$\iint_F |f'(z)|^2\,dx\,dy < \infty. \tag{2.4}$$

The Fuchsian group Γ is of the second kind if the set of limit points on ∂D is of measure zero; this is true if and only if D/Γ is a bordered Riemann surface. In particular, our group is of convergence type.

We give first an outline of our construction. Let

$$B = \bigcup_{n=1}^{\infty} \{0 \leq \operatorname{Re} w < 3^{-n},\ 2^n < \operatorname{Im} w < 2^{n+1}\} \cup \bigcup_{n=2}^{\infty} \{0 < \operatorname{Re} w < 3^{-n},\ \operatorname{Im} w = 2^n\}. \tag{2.5}$$

We attach a different copy of B suitably translated to each vertical side of B. Then we attach a new translated copy to each free vertical side of the resulting surface, and so on. We obtain a simply connected Riemann surface R that contains schlicht domains over each horizontal strip

$$(2.6) \qquad V_n = \{2^n < \operatorname{Im} w < 2^{n+1}\} \qquad (n = 1,2,\cdots)$$

of width 2^n. Hence the Lemma shows that the function $f(z)$ mapping D onto R is not a Bloch function. Furthermore R is invariant under a group of "translations" which corresponds to a Fuchsian group Γ in D, and some fundamental domain F of Γ is mapped one-to-one onto B. Hence (2.4) is satisfied because area $B < \infty$.

Proof. Let T be the free group generated by $(\sigma_n)_{n=1,2,\cdots}$. Thus each element of T can be uniquely written as a reduced word

$$(2.7) \qquad \tau = \sigma_{n_1}^{k_1}\cdots\sigma_{n_t}^{k_t} \qquad (k_\nu = \pm 1;\ \ k_{\nu+1} \neq -k_\nu \ \text{ if } \ n_\nu = n_{\nu+1})$$

where t is the length of τ. We define R as the disjoint union

$$(2.8) \qquad R = \bigcup_{\tau\in T} (B,\tau)$$

of "copies" of B. Every element $\tilde{w} \in R$ can be uniquely written as $\tilde{w} = (w,\tau)$ with $w \in B$ and τ of the form (2.7), and we define

$$(2.9) \qquad p(\tilde{w}) = w + \sum_{\nu=1}^{t} k_\nu 3^{-n_\nu} \qquad (\tilde{w} \in R).$$

Given $\tilde{w}_0 = (w_0,\tau) \in R$, we set $D_0 = \{|w - w_0| < \delta\}$ ($\delta > 0$ sufficiently small) and define a neighbourhood of $\tilde{w}_0$ by (D_0,τ) if $\operatorname{Re} w_0 \neq 0$, and by

$$(2.10) \qquad (D_0 \cap B,\tau) \cup ([D_0 + 3^{-n}] \cap B,\tau\sigma_n^{-1})$$

if $\operatorname{Re} w_0 = 0$, $2^n < \operatorname{Im} w_0 < 2^{n+1}$. Then $p(\tilde{w})$ maps each of our neighbourhoods one-to-one onto a disk in $\mathbb{C}$, and R together with the global parameter p is a Riemann surface which we denote again by R.

We show now that R is simply connected. Let C be a (piecewise smooth) closed curve on R; we may assume that C intersects (B,id). Let τ be a reduced word (of the form (2.7)) of maximal length such that C crosses the boundary $\partial(B,\tau)$ of (B,τ). Then C crosses $\partial(B,\tau)$ only on points on $\partial(B,\tau')$ where $\tau' = \sigma_{n_1}^{k_1}\cdots\sigma_{n_{t-1}}^{k_{t-1}}$. Since B is simply connected and $\partial(B,\tau) \cap \partial(B,\tau')$ is connected we can find a curve C' homotopic to C that does not cross $\partial(B,\tau)$. Now the reduced word τ' has length $t-1$. Repeating this process we see that C is homotopic to a curve in (B,id) and therefore to a point.

Given $\lambda \in T$ we define a conformal selfmapping λ^* of R by

$$\tilde{w} = (w,\tau) \rightarrow \lambda^*(\tilde{w}) = (w,\lambda\tau).$$

These mappings form a group T^* isomorphic to T. Also, by (2.9),

$$(2.11) \qquad p \circ \lambda^*(\tilde{w}) = p(w) + \sum_{\nu=1}^{\ell} j_\nu 3^{-m_\nu} \quad (\lambda = \sigma_{m_1}^{j_1}\cdots\sigma_{m_\ell}^{j_\ell}).$$

Since R is simply connected and has free boundary arcs there is a function $h(z)$ that maps D conformally and one-to-one onto R. If $\lambda^* \in T^*$ then

$$(2.12) \qquad \phi = h^{-1} \circ \lambda^* \circ h$$

maps D one-to-one onto D and is analytic, hence a Moebius transformation. Thus

$$\Gamma = \{\phi : \lambda^* \in T^*\}$$

is a Fuchsian group with fundamental domain $F = h^{-1}(\overset{\circ}{B},id)$. Since F has free boundary arcs on ∂D the group Γ is of the second kind.

The complex-valued function $f = p \circ h$ is analytic in D and satisfies, by (2.12) and (2.11),

$$f \circ \phi = p \circ \lambda^* \circ h = p \circ h + c = f + c$$

where $c = c(\phi)$ is a constant. Thus (2.3) is satisfied. By (2.9), $f(z)$ maps some domain in D one-to-one onto the strip V_n defined in (2.6). Since width $V_n = 2^n \to \infty$, the Lemma shows that $f(z)$ is not a Bloch function. Finally $f(z)$ maps F one-to-one onto B, and it follows that

$$\iint_F |f'(z)|^2 dx\, dy = \text{area}\, B = \sum_{n=1}^{\infty} (\tfrac{2}{3})^n < \infty.$$

3. AN INCLUSION RELATION

We prove now a different version of (1.3) (with $q = 1$). I want to thank the referee for his helpful comments.

THEOREM 3. *Let* Γ *be a Fuchsian group. Let* $f(z)$ *be analytic in* D *and*

(3.1) $$f(\phi(z)) = f(z) + c(\phi) \qquad (\phi \in \Gamma).$$

If, for some fundamental domain F,

(3.2) $$d = \text{diam}\, f(F) < \infty$$

then $f(z)$ *is a Bloch function.*

Our new condition (3.2) neither implies nor is implied by condition (2.4). It is not clear whether Theorem 3 has an analogue for $q \geq 2$. The quantity $\text{diam}\, f(F)$ depends on the choice of F but remains unchanged if we replace F by $\phi(F)$ $(\phi \in \Gamma)$.

Proof. Suppose that $f(z)$ is not a Bloch function. Then, by the Lemma, there exists a domain $\tilde{H} \subset D$ mapped homeomorphically by

$f(z)$ onto a disk $\{|w - w_0| < 5d\}$. Let

$$H = f^{-1}(\{|w - w_0| < 4d\}) \cap \tilde{H}.$$

Note that $\overline{H}$, the relative closure of H in D, is compact and homeomorphic to $\{|w - w_0| \leq 4d\}$ under $f(z)$. Also, if $K \subset D$ is connected and satisfies

$$(3.3) \qquad K \cap H \neq \emptyset \qquad\qquad f(K) \subset \{|w - w_0| \leq 3d\}$$

then $K \subset H$. For otherwise, K must contain a boundary point of H, and so $f(K)$ contains a point of $\partial f(H) = \{|w - w_0| = 4d\}$, contradicting (3.3).

Let z_0 be the preimage of w_0 in H, and let F be a fundamental domain for Γ which satisfies (3.2) and $z_0 \in \overline{F}$. Clearly

$$(3.4) \qquad f(\overline{F}) \subset \{|w - w_0| \leq d\}.$$

Thus $\overline{F}$ satisfies (3.3), and we conclude that $\overline{F} \subset H$.

Let $\phi_1(F)$ $(\phi_1 \in \Gamma)$ be a fundamental domain adjoining F. By (3.1), (3.2), and (3.4) we see that $f(\phi_1(\overline{F})) \subset \{|w - w_0| \leq 2d\}$; and (3.3) then yields $\phi_1(\overline{F}) \subset H$. Since $f(z)$ is univalent on H, we must have $c(\phi_1) \neq 0$. Let

$$S = \bigcup_{n \in Z} f(\phi^n(\overline{F})) = \bigcup_{n \in Z} [f(\overline{F}) + nc(\phi_1)].$$

Then S lies within an infinite strip of width at most d and so cannot cover $\{|w - w_0| < 4d\} = f(H)$. Since $f(z)$ is a homeomorphism on H, we can find another fundamental domain $\phi_2(F)$ $(\phi_2 \in \Gamma)$ adjoining F for which $f(\phi_2(\overline{F})) \not\subset S$. As before, $c(\phi_2) \neq 0$; and we clearly have

$$(3.5) \qquad \operatorname{Im}\, [c(\phi_1) \,/\, c(\phi_2)] \neq 0.$$

Because $\phi_1(F)$ and $\phi_2(F)$ adjoin F, the set

$$A = \bar{F} \cup \phi_1(\bar{F}) \cup \phi_2(\bar{F}) \cup \phi_1 \circ \phi_2(\bar{F}) \cup \phi_2 \circ \phi_1(\bar{F})$$

is connected. A moment's thought reveals that $f(A) \subset \{|w - w_0| \leq 3d\}$. Thus, by (3.3), $A \subset H$. Since $f(z)$ is univalent in H, we conclude from

$$f(\phi_1 \circ \phi_2(z)) = f(z) + c(\phi_1) + c(\phi_2) = f(\phi_2 \circ \phi_1(z)) \qquad (z \in H)$$

that $\phi_1 \circ \phi_2(z) = \phi_2 \circ \phi_1(z)$ for $z \in H$. But then $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1$ by the identity theorem.

Thus the subgroup of Γ generated by ϕ_1 and ϕ_2 is abelian, hence cyclic [6; p.14]. It follows that its homomorphic image $\{n_1 c(\phi_1) + n_2 c(\phi_2) : n_1, n_2 \in Z\}$ is also cyclic, and thus contradicts (3.5).

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